

# A Global Uniqueness on Spherically Stratified Dielectric Medium in Time-Harmonic Maxwell Equation with Interior Transmission Eigenvalues

Lung-Hui Chen<sup>1</sup>

July 11, 2012

## Abstract

A set of regularly distributed transmission eigenvalues generates a density function. We use such a density function inversely determines the form of the indicator function. Using the entire function theory, we reduce an uniqueness problem with interior transmission eigenvalues induced by time-harmonic Maxwell equation to an uniqueness problem in entire function theory. In such an inverse problem, the definite integral of the square root of refraction index is the main parameter.

MSC:35P25/35R30/34B24/.

Keywords: Maxwell equation/inverse problem/non-self-adjoint Sturm-Liouville problem/ interior transmission eigenvalue /Cartwright's theory/Wilder's theorem.

## 1 Introduction and the Main Result

In this paper, we consider the time-harmonic Maxwell equation with non-absorbing refraction index in the following setting:

$$n(x) = n(r) > 0, \quad r = |x|, \quad \text{when } r \in [0, a]; \quad \Im n = 0; \quad n(r) = 1, \quad \text{when } r \geq a; \quad n \in \mathcal{C}^2(\mathbb{R}); \quad (1.1)$$

such that

$$\begin{cases} \nabla \times E_1 - ikH_1 = 0, & \nabla \times H_1 + ikn(r)E_1 = 0, & \text{in } B; \\ \nabla \times E_0 - ikH_0 = 0, & \nabla \times H_0 + ikE_0 = 0, & \text{in } B; \end{cases} \quad (1.2)$$

with boundary condition

$$\nu \times (E_1 - E_0), \nu \times (H_1 - H_0) = 0, \quad \text{on } \partial B, \quad (1.3)$$

where  $E_0, H_0$  is an electromagnetic Herglotz pair,  $B$  is an open ball of radius  $a$  in  $\mathbb{R}^3$  with exterior unit normal vector  $\nu$ . We will look for a non-trivial solution to this homogeneous electromagnetic interior transmission problem (1.2) and (1.3). For each  $k \in \mathbb{C}$  such that (1.2) and (1.3) has a set of non-trivial solution is called an interior transmission eigenvalues. We reduce such an electromagnetic interior transmission problem to the acoustic interior transmission problem:

$$\begin{cases} \Delta w + k^2 n(r)w = 0, & \text{in } B; \\ \Delta v + k^2 v = 0 & \text{in } B; \\ w = v, \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial r} & \text{on } \partial B, \end{cases} \quad (1.4)$$

where  $w, v \in \mathcal{C}^3(B)$ . To see this, we consider the following quantity

$$\begin{cases} E_1(x) := \nabla \times \{xw(x)\}; \\ H_1(x) := \frac{1}{ik} \nabla \times \{E_1(x)\}; \\ E_0(x) := \nabla \times \{xv(x)\}; \\ H_0(x) := \frac{1}{ik} \nabla \times \{E_0(x)\}, \end{cases} \quad (1.5)$$

<sup>1</sup>Department of Mathematics, National Chung Cheng University, 168 University Rd. Min-Hsiung, Chia-Yi County 621, Taiwan. Email: mr.lunghuichen@gmail.com; lhchen@math.ccu.edu.tw. Fax: 886-5-2720497.

from which one can obtain a set of solution to the electromagnetic interior transmission problem (1.2) and (1.3). We refer the induction to the Colton and Kress [4].

We need to consider the solutions  $w, v$  to (1.4) that are not spherically symmetric. Therefore, we look for non-trivial solutions  $w, v$  in the following form:

$$v(r, \theta) = a_l j_l(jr) P_l(\cos \theta); \quad (1.6)$$

$$w(r, \theta) = b_l \frac{y_l(r)}{r} P_l(\cos \theta), \quad (1.7)$$

where  $P_l$  is Legendre's polynomial,  $j_l$  is the spherical Bessel function of degree  $l$ ,  $a_l$  and  $b_l$  are constants to be determined and the function  $y_l$  is a solution of

$$y_l'' + (k^2 n(r) - \frac{l(l+1)}{r^2}) y_l = 0; \quad (1.8)$$

$$\lim_{r \rightarrow 0} \left\{ \frac{y_l(r)}{r} - j_l(kr) \right\} = 0. \quad (1.9)$$

for  $r > 0$  and  $y_l$  is continuous for  $r \geq 0$ . Moreover, as demonstrated in [4], we consider the non-spherically symmetric  $w, v$ . In such a magnetic problem, we are asked to consider  $l \geq 1$ . Furthermore, we see that (1.9) implies

$$y_l(0) = 0; y_l'(0) = 0. \quad (1.10)$$

We will show there exist a set of  $k \in \mathbb{C}$  with its maximal density and constants  $a_l = a_l(k)$ ,  $b_l = b_l(k)$ , such that (1.6) and (1.7) is a set of non-trivial solution to the interior transmission problem (1.4). Considering (1.5), we see that, for any such value of  $k$ , the set of the electric far field patterns is not complete in certain functional space. See the discussion in [4].

The interior transmission problem (1.4) and (1.5) admits a set of non-trivial solution  $v, w$  if there exists a set of non-trivial solutions  $a_l, b_l$  to the following homogenous system

$$b_l \frac{y_l(a)}{a} - a_l j_l(ka) = 0; \quad (1.11)$$

$$b_l \frac{d}{dr} \left( \frac{y_l(r)}{r} \right) \Big|_{r=a} - a_l k j_l'(ka) = 0. \quad (1.12)$$

Such a system admits a set of non-trivial solutions  $a_l, b_l$  if and only if the determinant

$$d_l(k) := \det \begin{pmatrix} \frac{y_l(a)}{a} & -j_l(ka) \\ \frac{d}{dr} \left( \frac{y_l(r)}{r} \right) \Big|_{r=a} & -k j_l'(ka) \end{pmatrix} = 0. \quad (1.13)$$

$d_l(k)$  is an entire even function. See [5].

In this paper, we consider

$$l = 1; \quad (1.14)$$

$$d_1(k) := D(k), \quad k \in \mathbb{C}; \quad (1.15)$$

$$y_1(x, k) := y(x, k), \quad k \in \mathbb{C}. \quad (1.16)$$

May we ask that if the set of the interior transmission eigenvalues of (1.2), in particular, the set of interior transmission eigenvalues of the acoustic system (1.4) or zeros of  $D(k)$ , can uniquely determine the refraction index  $n(r)$ ?

Following the local uniqueness results in [12, 13], we state the uniqueness result in this paper.

**Theorem 1.1** *Let the functional determinant  $D(z)$  be defined as in (1.13) and (1.15). Then, the zeros of  $D(z)$  inside any of the angular wedges*

$$\Sigma_1 := \{k \in \mathbb{C} \mid -\epsilon \leq \arg k \leq \epsilon\}, \quad (1.17)$$

$$\Sigma_2 := \{k \in \mathbb{C} \mid \pi - \epsilon \leq \arg k \leq \pi + \epsilon\}, \quad \forall \epsilon > 0, \quad (1.18)$$

*uniquely determine between the spherical refraction indices  $n(r)$  if they have the same value at  $r = 0$ .*

## 2 Preliminaries: Cartwright's Theorem Versus Wilder's Theorem

From (1.13), we compute the  $D(k)$  as follows.

$$D(k) = \left(\frac{-k}{a}\right)y(a, k)j_1'(ka) + \frac{y'(a, k)}{a}j_1(ka) - \frac{1}{a^2}y(a, k)j_1(ka). \quad (2.1)$$

Since

$$j_1(t) = \frac{\sin t}{t^2} - \frac{\cos t}{t}, \quad (2.2)$$

$$\begin{aligned} D(k) &= \left(\frac{-k}{a}\right)y(a, k)\left[\frac{2\cos(ka)}{(ka)^2} + \left(\frac{1}{ka} - \frac{2}{(ka)^3}\right)\sin ka\right] \\ &\quad + \frac{y'(a, k)}{a}\left[\frac{\sin ka}{(ka)^2} - \frac{\cos ka}{ka}\right] - \frac{1}{a^2}y(a, k)\left[\frac{\sin ka}{(ka)^2} - \frac{\cos ka}{ka}\right] \\ &= \left(\frac{-k}{a}\right)y(a, k)\left[\frac{\cos ka}{(ka)^2} + \left(\frac{1}{ka} - \frac{1}{(ka)^3}\right)\sin ka\right] + \frac{y'(a, k)}{a}\left[\frac{\sin ka}{(ka)^2} - \frac{\cos ka}{ka}\right]. \end{aligned} \quad (2.3)$$

To study the asymptotics of  $D(k)$ , we study the asymptotics of the fundamental solutions  $y(a, k)$  and  $y'(a, k)$ . For this we use the asymptotic expansion theory for Sturm-Liouville problem provided in Erdelyi [8] on page 84 with notations there.

$$q(r, k) = n(r)k^2 - \frac{2}{r^2}, \quad (2.4)$$

which is considered as a power series in  $\frac{1}{k}$ . In particular, we set

$$q_0(r) = n(r), \quad q_1(r) = 0, \quad q_2(r) = -\frac{2}{r^2}. \quad (2.5)$$

Surely we have  $n(r) > 0$ ,  $\forall r > 0$ , but  $q_2(r)$  is singular at  $r = 0$ . Moreover, to fulfill the requirements in [8], we use

$$N = 1, \quad S := \{k \in \mathbb{C} \mid \Re\{k[-q_0(r)]^{\frac{1}{2}}\} \neq 0\}. \quad (2.6)$$

In this paper, we take

$$S = \{k \in \mathbb{C} \mid 0 < \arg k < \pi\} \cup \{k \in \mathbb{C} \mid \pi < \arg k < 2\pi\}.$$

Accordingly, we have a set of fundamental solutions  $y_1(r)$ ,  $y_2(r)$  to the problem (1.8) such that in the sectorial region  $S$

$$y_j(r; k) = Y_j(r)[1 + O(\frac{1}{k})]; \quad (2.7)$$

$$y_j'(r; k) = Y_j'(r)[1 + O(\frac{1}{k})], \quad (2.8)$$

as  $k \rightarrow \infty$  in  $S$ , uniformly for  $\delta \leq r \leq a$  and for  $\arg k$ , where  $\delta$  is arbitrarily small and

$$Y_j(r) = \exp\{\beta_{0j}k + \beta_{1j} + \beta_{2j}\frac{1}{k}\}, \quad (2.9)$$

where  $\beta_{0j}$ ,  $\beta_{1j}$  and  $\beta_{2j}$  satisfy

$$(\beta_{0j}')^2 + n(r) = 0; \quad (2.10)$$

$$2\beta_{0j}'\beta_{1j}' + \beta_{0j}'' = 0; \quad (2.11)$$

$$2\beta_{0j}'\beta_{2j}' - \frac{2}{r^2} + (\beta_{1j}')^2 + \beta_{1j}'' = 0. \quad (2.12)$$

Following from (2.5),

$$\beta_{0j}(r) = \pm i \int_{0+}^r \sqrt{q_0(\rho)} d\rho + C_{0j}; \quad (2.13)$$

$$\beta_{1j}(r) = -\frac{1}{4} \ln[q_0(r)] + C_{1j}; \quad (2.14)$$

$$\beta_{2j}(r) = \int_{0+}^r \frac{\frac{2}{\rho^2} - (\beta'_{1j}(\rho))^2 - \beta''_{1j}(\rho)}{2\beta'_{0j}(\rho)} d\rho + C_{2j} \quad (2.15)$$

$$= \mp i \int_{0+}^r \frac{\frac{2}{\rho^2} - (\beta'_{1j}(\rho))^2 - \beta''_{1j}(\rho)}{2\sqrt{n(\rho)}} d\rho + C_{2j} \quad (2.16)$$

$$:= \pm i \beta_2(r) + C_{2j}; \quad (2.17)$$

where  $C_{0j}$ ,  $C_{1j}$ ,  $C_{2j}$  are constants. Since  $n(r) > 0$  over  $[0, a]$  and is in  $\mathcal{C}^2(\mathbb{R})$ ,  $\beta_{0j}(r)$ ,  $\beta_{1j}(r)$ ,  $\beta_{2j}(r)$  are all sufficiently smooth and bounded over  $[0, a]$ . That is we can set

$$\beta_2(r) = - \int_{0+}^r \frac{\frac{2}{\rho^2} - (\beta'_{1j}(\rho))^2 - \beta''_{1j}(\rho)}{2\sqrt{n(\rho)}} d\rho. \quad (2.18)$$

In addition, we see

$$\beta'_{2j}(a) = \mp i \frac{1}{a^2}; \quad (2.19)$$

$$\beta'_{1j}(r) = -\frac{1}{4} \frac{q'_0(r)}{q_0(r)}; \quad (2.20)$$

$$\beta''_{1j}(r) = -\frac{1}{4} \frac{q''_0(r)q_0(r) - (q'_0(r))^2}{q_0^2(r)}. \quad (2.21)$$

Therefore,

$$\beta_2(r) = - \int_0^r \frac{\frac{2}{\rho^2} - (\beta'_{1j}(\rho))^2 - \beta''_{1j}(\rho)}{2\sqrt{n(\rho)}} d\rho. \quad (2.22)$$

In general, for  $n \geq 3$ ,  $j = \pm 1$ ,

$$\beta'_{nj}(r) = \frac{-\sum_{m=1}^{n-1} \beta'_{mj}(r)\beta'_{n-mj}(r) - \beta''_{n-1j}(r)}{2\beta'_{0j}(r)}; \quad (2.23)$$

$$\beta_{nj}(r) = \frac{\pm i}{2} \int_0^r \frac{\sum_{m=1}^{n-1} \beta'_{mj}(\rho)\beta'_{n-mj}(\rho) + \beta''_{n-1j}(\rho)}{\sqrt{n(\rho)}} d\rho. \quad (2.24)$$

Now we use (2.13), (2.14) and (2.17). The general solution to (1.8) is of the form

$$y(r) = [\alpha \frac{1}{[q_0(r)]^{\frac{1}{4}}} e^{(i \int_0^r \sqrt{q_0(\rho)} d\rho)k + i\beta_2(r)\frac{1}{k}} + \beta \frac{1}{[q_0(r)]^{\frac{1}{4}}} e^{(-i \int_0^r \sqrt{q_0(\rho)} d\rho)k - i\beta_2(r)\frac{1}{k}}][1 + O(\frac{1}{k})]. \quad (2.25)$$

We consider  $\alpha = \alpha(k)$ ,  $\beta = \beta(k)$ . To find such a set of constants, we compare (2.25) with its asymptotic behavior on  $0i + \mathbb{R}$ . For this, we combine from [4, p. 263] with an errata in [5, p.3] that

$$y(r) = \frac{1}{k[n(r)]^{\frac{1}{4}}} \cos(k \int_0^r [n(\rho)]^{\frac{1}{2}} d\rho - \pi) + O(\frac{\ln k}{k^2}). \quad (2.26)$$

That is  $\alpha = \frac{1}{2k} e^{-i\pi}$ ;  $\beta = \frac{1}{2k} e^{i\pi}$ . Since  $y(r, k)$  is an entire function in  $k$ , (2.25) and (2.26) combine to give

$$y(r, k) = \frac{1}{k[n(r)]^{\frac{1}{4}}} \cos[k \int_0^r [n(\rho)]^{\frac{1}{2}} d\rho - \pi + \frac{\beta_2(r)}{k}][1 + O(\frac{1}{k})], \quad \mathbb{C} \setminus 0i + \mathbb{R}, \quad (2.27)$$

uniformly in  $r, \arg k$ . Hence, (2.8) gives

$$\begin{aligned} y'(r, k) &= \left\{ -[n(r)]^{\frac{1}{2}} - \frac{\beta_2'(r)}{k^2 n(r)^{\frac{1}{4}}} \right\} \sin \left[ k \int_0^r [n(\rho)]^{\frac{1}{2}} d\rho - \pi + \frac{\beta_2(r)}{k} \right] [1 + O(\frac{1}{k})] \\ &\quad + \frac{-1}{4k} n(r)^{-\frac{5}{4}} n'(r) \cos \left[ k \int_0^r [n(\rho)]^{\frac{1}{2}} d\rho - \pi + \frac{\beta_2(r)}{k} \right] [1 + O(\frac{1}{k})], \end{aligned} \quad (2.28)$$

in  $\mathbb{C} \setminus 0i + \mathbb{R}$ .

$$y'(a, k) = [-1 + \frac{1}{a^2 k^2}] \sin \left[ k \int_0^a [n(\rho)]^{\frac{1}{2}} d\rho - \pi + \frac{\beta_2(a)}{k} \right] [1 + O(\frac{1}{k})]. \quad (2.29)$$

Using this, we compute the functional determinant  $D(z)$  in (2.3) in  $\mathbb{C} \setminus 0i + \mathbb{R}$ . We obtain from (2.3), (2.27) and (2.28) that

$$\begin{aligned} D(z) &= \cos \left[ z \int_0^a [n(\rho)]^{\frac{1}{2}} d\rho - \pi + \frac{\beta_2(r)}{z} \right] \cos [za] \left[ -\frac{1}{a^3 z^2} \right] [1 + O(\frac{1}{z})] \\ &\quad + \cos \left[ z \int_0^a [n(\rho)]^{\frac{1}{2}} d\rho - \pi + \frac{\beta_2(r)}{z} \right] \sin [za] \left[ \frac{-1}{a^2 z} + \frac{1}{a^4 z^3} \right] [1 + O(\frac{1}{z})] \\ &\quad + \sin \left[ z \int_0^a [n(\rho)]^{\frac{1}{2}} d\rho - \pi + \frac{\beta_2(r)}{z} \right] \cos [za] \left[ \frac{1}{a^2 z} - \frac{1}{a^4 z^3} \right] [1 + O(\frac{1}{z})] \\ &\quad + \sin \left[ z \int_0^a [n(\rho)]^{\frac{1}{2}} d\rho - \pi + \frac{\beta_2(r)}{z} \right] \sin [za] \left[ \frac{-1}{a^3 z^2} + \frac{1}{a^5 z^4} \right] [1 + O(\frac{1}{z})] \end{aligned} \quad (2.30)$$

$$\begin{aligned} &= \frac{-1}{a^3 z^2} \left[ \cos z(a - \int_0^a [n(\rho)]^{\frac{1}{2}} d\rho) + \pi - \frac{\beta_2(r)}{z} \right] [1 + O(\frac{1}{z})] \\ &\quad + \left[ -\frac{1}{a^2 z} + \frac{1}{a^4 z^3} \right] \left[ \sin z(a - \int_0^a [n(\rho)]^{\frac{1}{2}} d\rho) + \pi - \frac{\beta_2(r)}{z} \right] [1 + O(\frac{1}{z})] \\ &\quad + \frac{\sin \{ z \int_0^a [n(\rho)]^{\frac{1}{2}} d\rho - \pi + \frac{\beta_2(r)}{z} \} \sin [za]}{a^5 z^4} [1 + O(\frac{1}{z})]. \end{aligned} \quad (2.31)$$

We set

$$N(a) := \int_0^a [n(\rho)]^{\frac{1}{2}} d\rho. \quad (2.32)$$

Therefore, we have

$$\begin{aligned} D(z) &= \frac{-1}{a^3 z^2} \left[ \cos za - zN(a) + \pi - \frac{\beta_2(r)}{z} \right] [1 + O(\frac{1}{z})] \\ &\quad + \left[ -\frac{1}{a^2 z} + \frac{1}{a^4 z^3} \right] \left[ \sin za - zN(a) + \pi - \frac{\beta_2(r)}{z} \right] [1 + O(\frac{1}{z})] \\ &\quad + \frac{\sin \{ zN(a) - \pi + \frac{\beta_2(r)}{z} \} \sin [za]}{a^5 z^4} [1 + O(\frac{1}{z})]. \end{aligned} \quad (2.33)$$

Actually, it has been shown in [4] if considered with the errata in [5, p.3] that

$$d_l(z)|_{l=1} = \frac{1}{a^2 z} \left\{ \sin \{ z(N(a) - a) - \pi \} + O(\frac{\ln z}{z}) \right\}, \quad z \in 0i + \mathbb{R}. \quad (2.34)$$

We see (2.33) matches with (2.34) when limited on  $0i + \mathbb{R}$ . Without loss of generality, we discuss the zeros of

$$z^4 D(z) := D_1(z) + D_2(z) + D_3(z), \quad (2.35)$$

where  $D_i(z)$  is the  $i$ -th term in (2.33) with a multiple of  $z^4$ ,  $i = 1, 2, 3$ . For this representation form of an entire function, we consider the value distribution theory from complex analysis which we refer to the Levin's book [10, 11] and Cartwright's theory [2, 3]. Let us review the following definitions.

**Definition 2.1** Let  $f(z)$  be an entire function. Let  $M_f(r) := \max_{|z|=r} |f(z)|$ . An entire function  $f(z)$  is said to be a function of finite order if there exists a positive constant  $k$  such that the inequality

$$M_f(r) < e^{r^k} \quad (2.36)$$

is valid for all sufficiently large values of  $r$ . The greatest lower bound of such numbers  $k$  is called the order of the entire function  $f(z)$ . By the type  $\sigma$  of an entire function  $f(z)$  of order  $\rho$ , we mean the greatest lower bound of positive number  $A$  for which asymptotically we have

$$M_f(r) < e^{Ar^\rho}. \quad (2.37)$$

That is

$$\sigma := \limsup_{r \rightarrow \infty} \frac{\ln M_f(r)}{r^\rho}. \quad (2.38)$$

If  $0 < \sigma < \infty$ , then we say  $f(z)$  is of normal type or mean type.

We also see that

$$e^{(\sigma-\epsilon)r^\rho} < \liminf_n M_f(r) < \limsup_{\text{as}} e^{(\sigma+\epsilon)r^\rho}, \quad (2.39)$$

where the first inequality holds for some sequence going to infinity and the second one holds asymptotically.

**Definition 2.2** If an entire function  $f(z)$  is of order one and of normal type, then we say it is an entire function of exponential type  $\sigma$ .

**Definition 2.3** Let  $\rho \in \mathbb{R}$  and  $\rho(r) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ . We say  $\rho(r)$  is a proximate order to  $\rho$  if

$$\lim_{r \rightarrow \infty} \rho(r) = \rho \geq 0; \quad \lim_{r \rightarrow \infty} r \rho'(r) \ln r = 0. \quad (2.40)$$

**Definition 2.4** Let  $f(z)$  be an integral function of finite order in the angle  $[\theta_1, \theta_2]$ . We call the following quantity as the generalized indicator of the function  $f(z)$ .

$$h_f(\theta) := \limsup_{r \rightarrow \infty} \frac{\ln |f(re^{i\theta})|}{r^{\rho(r)}}, \quad \theta_1 \leq \theta \leq \theta_2, \quad (2.41)$$

where  $\rho(r)$  is some proximate order.

The order and the type of an integral function in an angle can be defined similarly. The connection between the indicator  $h_f(\theta)$  and its type  $\sigma$  is specified by the following theorem.

**Lemma 2.5 (Levin [10], p.72)** The maximum value of the indicator  $h_f(\theta)$  of the function  $f(z)$  on the interval  $\alpha \leq \theta \leq \beta$  is equal to the type  $\sigma_f$  of this function inside the angle  $\alpha \leq \arg z \leq \beta$ .

**Lemma 2.6** Let  $A, B, C$  and  $D$  be real constants.

$$h_{\cos\{Az+B\}}(\theta) = |A \sin \theta|; \quad (2.42)$$

$$h_{\sin\{Cz+D\}}(\theta) = |C \sin \theta|. \quad (2.43)$$

**Proof** We observe that

$$|\cos Az + B|^2 = \sinh^2\{Ay\} + \cos^2\{Ax + B\}; \quad (2.44)$$

$$|\sin Cz + D|^2 = \sinh^2\{Cy\} + \sin^2\{Cx + D\}. \quad (2.45)$$

Applying definition (2.41), we prove the lemma.  $\square$

**Lemma 2.7** For any entire function  $D(z)$ , the following identity holds.

$$h_{z^4 D(z)}(\theta) = h_{D(z)}(\theta).$$

**Proof**  $z^4D(z)$  and  $D(z)$  are both entire function. We see that

$$h_{z^4D(z)}(\theta) = \limsup_{r \rightarrow \infty} \frac{\ln |z^4D(z)|}{r} = \limsup_{r \rightarrow \infty} \frac{4 \ln |z| + \ln |D(z)|}{r} = h_{D(z)}(\theta). \quad (2.46)$$

□

We mention two more inequalities for indicator functions.

**Lemma 2.8** *Let  $f, g$  be two entire functions. Then, the following two inequalities hold.*

$$h_{fg}(\theta) = h_f(\theta) + h_g(\theta), \text{ if one limit exists;} \quad (2.47)$$

$$h_{f+g} \leq \max_{\theta} \{h_f(\theta), h_g(\theta)\}, \quad (2.48)$$

where if the indicator of the two summands are not equal at some  $\theta_0$ , then the equality holds in (2.48).

**Proof** We can find these in [10, p.51]. □

**Lemma 2.9** *If  $a \neq \int_0^a \sqrt{n(\rho)} d\rho$ , then the indicator function of  $D_3(z)$  in (2.35)*

$$h_{D_3}(\theta) = (a + N(a)) |\sin \theta|; \quad (2.49)$$

$$h_{D_1}(\theta) = |N(a) - a| |\sin \theta|;$$

$$h_{D_2}(\theta) = |N(a) - a| |\sin \theta|;$$

$$h_{z^4D}(\theta) = (a + N(a)) |\sin \theta|. \quad (2.50)$$

If  $a = \int_0^a \sqrt{n(\rho)} d\rho$ , then

$$h_{z^4D}(\theta) = 2a |\sin \theta|. \quad (2.51)$$

**Proof** The indicator function of  $\cos\{zN(a)\}$  is  $|N(a)| |\sin \theta|$  and the one for  $\sin za$  is  $a |\sin \theta|$ . Applying (2.41) and (2.47) to  $D_3(z)$ , we prove the (2.49). Similarly, we use (2.42) and (2.43) to prove the  $h_{D_1}(\theta)$  and  $h_{D_2}(\theta)$  respectively. Accordingly, we apply (2.48) to obtain

$$\begin{aligned} h_{D_1+D_3}(\theta) &= \max_{\theta} \{h_{D_1}(\theta), h_{D_3}(\theta)\} \\ &= \max_{\theta} \{|N(a) - a| |\sin \theta|, |N(a) + a| |\sin \theta|\} \end{aligned} \quad (2.52)$$

$$= |N(a) + a| |\sin \theta|. \quad (2.53)$$

Finally, using (2.47) and (2.48) again,

$$\begin{aligned} h_D(\theta) &= h_{D_1+D_2+D_3}(\theta) \\ &= \max_{\theta} \{h_{D_1+D_3}(\theta), h_{D_2}(\theta)\} \end{aligned} \quad (2.54)$$

$$= (a + N(a)) |\sin \theta|. \quad (2.55)$$

To prove (2.51), we begin with (2.33).

$$\begin{aligned} h_{z^4D_3}(\theta) &= \lim_{r \rightarrow \infty} \frac{\ln |\sin\{zN(a)\} \sin\{za\}|}{r} \\ &= \lim_{r \rightarrow \infty} \frac{\ln |\{\cos z(N(a) - a) - \cos z(N(a) + a)\}/2|}{r}. \end{aligned} \quad (2.56)$$

Since  $a = \int_0^a \sqrt{n(\rho)} d\rho$ , (2.51) follows from (2.42). □

**Definition 2.10** *We say an entire function is of class  $C$  if the following two conditions hold:*

$$f(z) \text{ is an entire function of exponential type;} \quad (2.57)$$

$$\int_{-\infty}^{\infty} \frac{\log^+ |f(t)|}{1+t^2} dt < \infty. \quad (2.58)$$

**Lemma 2.11** *The entire function  $z^4D(z)$  in (2.35) is an entire function of class C.*

**Proof** The condition (2.58) is direct from (2.34). From (1.13), each term there is an entire function of normal type. Hence,  $z^4D(z)$  is of class C.  $\square$

We need more tools to understand the distribution of the zeros of  $z^4D(z)$  and its analytic behavior.

**Definition 2.12** *The following quantity is called the length of the indicator diagram of entire function  $f$ :*

$$d = h_f\left(\frac{\pi}{2}\right) + h_f\left(-\frac{\pi}{2}\right). \quad (2.59)$$

**Lemma 2.13** *If (2.57) and (2.58) hold, then  $f(z)$  is of completely regular growth and all of the zeros of  $f(z)$ , except possibly a set of zero density, lie inside arbitrarily small angles  $|\arg z| < \epsilon$  and  $|\arg z - \pi| < \epsilon$ , where the density*

$$\Delta_i := \lim_{t \rightarrow \infty} \frac{n_i(t)}{t} \quad (2.60)$$

*of the set of the zeros within each of these angles is equal to  $d/2\pi$ , where  $d$  is the length of the indicator diagram,  $i = 1, 2$  denotes the quantity corresponding to each of the two such angles along the real axis and  $n_i(t)$  is the number of zeros in each angles with norm less than  $t$ .*

This theorem is in Levin [10, p.251].

**Lemma 2.14** *The length of the indicator diagram of  $z^4D(z)$  is  $2(a + N(a))$ . The density in each of the two small angles along real axis is  $(a + N(a))/\pi$ .*

**Proof** This follows from (2.50) and definition (2.59).  $\square$

Such a density theorem is crucial in this paper. We present an alternative justification from the point of view of the zeros of exponential sums. Let's discuss Wilder's type of theorem for exponential sums. We refer to the work of D.G. Dickson [6, 7] and R.E. Langer [9]. The reference [6] is a concise review and [7] is a detailed lecture. The reference [9] is a step-by-step introduction to such a theory.

Let

$$f(z) = \sum_{j=1}^n A_j z^{m_j} [1 + \epsilon(z)] e^{\omega_j z}, \quad (2.61)$$

where  $n > 1$  and  $A_j$  and  $\omega_j$  are complex numbers such that  $A_j \neq 0$  and the  $\omega_j$  are distinct; the  $m_j$  are non-negative integers; the functions  $\epsilon$  are analytic for  $|z| \geq r_0 \geq 0$  with  $\lim_{z \rightarrow \infty} \epsilon(z) = 0$ . When we are talking about the zeros of  $f(z)$ , we are referring to its zeros outside certain open ball around the origin. A function of the form

$$\sum_{j=1}^n P_j(z) e^{\omega_j z}, \quad (2.62)$$

where  $P_j(z)$  are polynomials, is a straightforward example of (2.61). Our  $z^4D(z)$  in (2.35) would yield another special case:

$$z^4D(z) = [c_1 e^{-iN(a)z - iaz} + c_2 z^3 e^{iN(a)z - iaz} + c_3 z^3 e^{iaz - iN(a)z} + c_4 e^{iN(a)z + iaz}] [1 + O(\frac{1}{z})], \quad (2.63)$$

where  $c_2, c_3$  asymptotically approach to constants as  $z \rightarrow \infty$  and  $c_1, c_4$  are constants.

An introductory version of Wilder's theorem is of the following form.

**Theorem 2.15 (Dickson)** *Let*

$$R(\alpha, s, H) := \{z = x + iy \in \mathbb{C} \mid |x| \leq H, y \in [\alpha, \alpha + s]\}; \quad (2.64)$$

$$N_g(R(\alpha, s, H)) := \{ \text{the numbers of } g(z) \text{ in } R(\alpha, s, H) \}. \quad (2.65)$$

*Let  $g(z) = \sum_{j=1}^n A_j e^{\omega_j z}$ , where  $z = x + iy$ ,  $A_j \neq 0$ ,  $\omega_1 < \omega_2 < \dots < \omega_n$ . Then, there exists  $K > 0$  such that*



1. each zero of  $g$  is in  $|x| < K$ ;
2. for each pair of reals  $(\alpha, s)$  with  $s > 0$ ,

$$|N_g(R(\alpha, s, K)) - s(\omega_n - \omega_1)/(2\pi)| \leq n - 1. \quad (2.66)$$

We refer these to [6].

To consider our modified determinant  $z^4 D(z)$ , we need a more sophisticated version of such a theorem. We set up the following quantities to the  $f(z)$  in (2.61): let  $Q$  be the broken line given by the  $\bar{\omega}_j$  given in (2.61) with  $\bar{\omega}_1, \dots, \bar{\omega}_\sigma$  as its vertices. The indices is labelled counterclockwise. Let  $L_k$  be the line segment  $[\bar{\omega}_k, \bar{\omega}_{k+1}]$  and

$$\phi_k := \arg\{\bar{\omega}_k - \bar{\omega}_{k+1}\}$$

in  $[-\frac{\pi}{2}, \frac{3\pi}{2})$ . Let

$$e_k = e^{i\phi_k}.$$

Certain  $\bar{\omega}_p$  on  $L_k$  are assigned doubly indexed subscripts as follows: let the convex hull of  $\bar{\omega}_k, \bar{\omega}_{k+1}$  and  $\tau_p = \bar{\omega}_p + im_p e_k$  in which  $\bar{\omega}_p$  on  $L_k$ ; assign subscripts  $j = 1, \dots, \sigma_k$  to  $\omega_{kj}$  so that  $\omega_{k1} = \omega_k, \omega_{k\sigma_k} = \omega_{k+1}$  and  $\tau_{kj}$  are vertices of this convex hull and preceding in a counterclockwise direction from  $\bar{\omega}_k + im_k e_k$  to  $\bar{\omega}_{k+1} + im_{k+1} e_k$ . For  $j = 1, \dots, \sigma_k - 1$ ,

$$\begin{aligned} L_{kj} &:= [\tau_{kj}, \tau_{kj+1}]; \\ \mu_{kj} &:= \frac{m_{kj} - m_{kj+1}}{(\omega_{kj} - \omega_{kj+1})e_k} \end{aligned} \quad (2.67)$$

which is real;  $n_{kj}$  is the number of  $\tau_p$  on  $L_{kj}$ .

Moreover, for  $j = 1, \dots, \sigma_k - 1$  and  $H > 0$ , we define

$$V_{kj}(H) := \{z | \Im(z/e_k) \geq 0, |\Re(z/e_k) + \mu_{kj} \log |z|| \leq H\}; \quad (2.68)$$

$T_k(\theta)$  is defined to be a closed sector with vertex at zero of opening  $2\theta$  about the outward normal to  $L_k$  through the origin. For the same  $k$  and  $j$  and each triple of reals  $(\alpha, s, H)$ ,  $s > 0$  and  $H > 0$ , the set

$$R_{kj}(\alpha, s, H) := \{z | \Im(z/e_k) + \mu_{kj} \arg z \in [\alpha, \alpha + s], |\Re(z/e_k) + \mu_{kj} \log |z|| \leq H\}, \quad (2.69)$$

where  $\arg z \in (\phi_k, \phi_k + \pi)$ . It's a curvilinear tubular neighborhood along some direction. We may collect a few facts from [6, 7].

**Lemma 2.16** *The following properties hold.*

1. The size of  $R_{kj}(\alpha, s, H)$  is approximately a rectangle of dimension  $s$  by  $2H$  for large  $\alpha$ .
2.  $V_{kj}$  is individually connected and disjoint to each other for each pair of  $(k, j)$  when  $|z|$  is large.
3. The boundaries of  $V_{kj}$  are logarithmic to the outward normal to  $L_k$  to the exterior of  $Q$ .
4. For each fixed  $\theta, H$ , the subsets of  $V_{kj}(H)$  are in  $T_k(\theta)$  for large  $z$ .
5. For fixed  $\theta$  and  $H$ ,

$$R_{kj}(\alpha, s, H) \subset V_{kj} \cap T_k(\theta), \forall s > 0, \text{ for large } \alpha. \quad (2.70)$$

Let us define

$$N_f(R_{kj}(\alpha, s, H)) := \{\text{the numbers of } f(z) \text{ in } R_{kj}(\alpha, s, H)\}, \quad (2.71)$$

where the zeros are counted according to their multiplicity. We refer to the page 4 and 10 in [7] for some delicate diagrams on all of these quantities. Most importantly, we have the following theorem.

**Theorem 2.17 (Dickson)** *Let  $f(z)$  be given as in (2.61). Then, there exists  $K > 0$  such that (a) all but a finite number of zeros of  $f$  of modulus greater than  $r_0$  are in  $\cup_{k,j} V_{kj}$  and (b) for each pair of positive reals  $\epsilon$  and  $s_0$ , there exists an  $\alpha_0 = \alpha_0(\epsilon, s_0)$  such that whenever  $\alpha \geq \alpha_0$  and  $s \geq s_0$ ,*

$$|N_f(R_{kj}(\alpha, s, K)) - s|\omega_{kj+1} - \omega_{kj}|/(2\pi)| < n_{kj} - 1 + \epsilon. \quad (2.72)$$

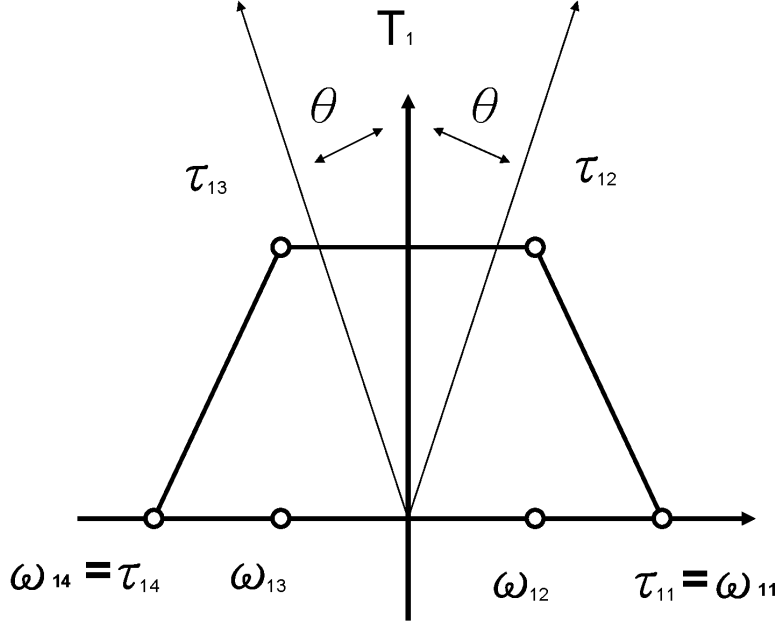


Figure 1: The polygon Q

This is stated as in [6].

Without loss of generality, we consider a Wilder's type of theorem to (2.63) in the following setting:

$$\mathcal{D}(z) := (z/i)^4 D(z/i) = c_1 e^{-N(a)z - az} + c_2' z^3 e^{N(a)z - az} + c_3' z^3 e^{az - N(a)z} + c_4 e^{N(a)z + az} \quad (2.73)$$

and

$$a > N(a). \quad (2.74)$$

We set

$$\omega_1 = \bar{\omega}_1 = a + N(a), \omega_2 = \bar{\omega}_2 = -a - N(a), \text{ as points in } \mathbb{C}. \quad (2.75)$$

Therefore, we do the double-indexing for  $\mathcal{D}(z)$  as follows.

$$\bar{\omega}_{11} = \tau_{11} = a + N(a), \bar{\omega}_{12} = a - N(a), \bar{\omega}_{13} = -a + N(a), \bar{\omega}_{14} = \tau_{14} = -a - N(a). \quad (2.76)$$

We refer to Figure 1 below for a diagram. Furthermore, the line segment

$$L_1 = [\omega_{14}, \omega_{11}], L_{11} = [\omega_{11}, \tau_{12}], L_{12} := [\tau_{12}, \tau_{13}], L_{13} := [\tau_{13}, \omega_{14}]. \quad (2.77)$$

These are edges for polygon  $Q$ .

$$\phi_1 = \arg(\omega_1 - \omega_2) = 0. \quad (2.78)$$

Henceforth, we have

$$e_1 = 1. \quad (2.79)$$

By (2.73) and the definition of  $\tau_p$ ,

$$m_{11} = 0, m_{12} = 3, m_{13} = 3, m_{14} = 0. \quad (2.80)$$

Therefore,

$$\tau_{12} = a - N(a) + 3i; \tau_{13} := N(a) - a + 3i. \quad (2.81)$$

We compute more. From (2.67), we obtain

$$\mu_{11} = \frac{-3}{2N(a)}; \mu_{12} = 0; \mu_{13} = \frac{3}{2N(a)}. \quad (2.82)$$

Accordingly, we have

$$V_{11} = \{z | \Im(z) \geq 0, |\Re(z) + \frac{-3}{2N(a)} \log |z|| \leq H\}, \quad (2.83)$$

$$V_{12} = \{z | \Im(z) \geq 0, |\Re(z)| \leq H\}, \quad (2.84)$$

$$V_{13} = \{z | \Im(z) \geq 0, |\Re(z) + \frac{3}{2N(a)} \log |z|| \leq H\}. \quad (2.85)$$

Combing (2.73) to (2.85) and Theorem 2.17, we prove the following theorem.

**Theorem 2.18** *There exist some  $K > 0$  and large  $\alpha$  such that*

$$N_{\mathcal{D}}(R_{11}(\alpha, s, K)) \sim \begin{cases} s^{\frac{N(a)}{\pi}}, & a \geq N(a); \\ s^{\frac{a}{\pi}}, & a < N(a); \end{cases} \quad (2.86)$$

$$N_{\mathcal{D}}(R_{12}(\alpha, s, K)) \sim s^{\frac{|a - N(a)|}{\pi}}; \quad (2.87)$$

$$N_{\mathcal{D}}(R_{13}(\alpha, s, K)) \sim \begin{cases} s^{\frac{N(a)}{\pi}}, & a \geq N(a); \\ s^{\frac{a}{\pi}}, & a < N(a). \end{cases} \quad (2.88)$$

For the critical case  $a = N(a)$ , we have to reset the polygonal  $Q$ . We define:  $\tau_{11} = \bar{\omega}_1 = \bar{\omega}_{11}$ ;  $\bar{\omega}_{11} = \tau_{11} = 2a$ ,  $\bar{\omega}_{12} = 0$ ,  $\bar{\omega}_{13} = \tau_{13} = -2a$ . Accordingly,  $m_{11} = 0$ ,  $m_{12} = 3$ ,  $m_{13} = 0$ . Moreover,  $\mu_{11} = -\frac{3}{2a}$ ,  $\mu_{12} = \frac{3}{2a}$ . For a quick picture, we just delete (2.87) for the critical case.

**Corollary 2.19** *Let  $\Delta_i$ ,  $i = 1, 2$ , be defined as in Lemma 2.13. There exists some  $\alpha_0$  such that*

$$\Delta_i = N_{\mathcal{D}}(\cup_{j=1}^3 R_{1j}(\alpha, s, K)) = \frac{a + N(a)}{\pi}, \text{ whenever } \alpha \geq \alpha_0. \quad (2.89)$$

**Proof** Since  $\{V_{kj}\}$  are disjoint to each other for large  $\alpha$ , we sum up all three asymptotic behaviors from (2.86), (2.87) and (2.88) to conclude that

$$N_{\mathcal{D}}(\cup_{j=1}^3 R_{1j}(\alpha, s, K)) = \frac{a + N(a)}{\pi}. \quad (2.90)$$

Also,  $[\omega_{14}, \omega_{13}]$ ,  $[\omega_{13}, \omega_{12}]$ ,  $[\omega_{12}, \omega_{11}]$  have the same outward normal, so  $R_{11}(\alpha, s, K)$ ,  $R_{12}(\alpha, s, K)$  and  $R_{13}(\alpha, s, K)$  asymptotically fall in the same wedge  $T(\theta)$  around this outward normal with any fixed  $\theta$ . Let  $\theta$  be small. We prove the corollary.  $\square$

We see that (2.84) and (2.87) is weaker than the transmission eigenvalues distribution theory in [14]. However, (2.86) and (2.88) offer some windows outside the real axis.

### 3 Proof of Theorem 1.1

Let  $D^i(z)$  be the functional determinant corresponding to refraction index  $n^i(r)$ ,  $i = 1, 2$ . If the zeros of  $D^i$  in either wedge coincide, then Lemma 2.14 tells us that the indicator diagrams from the two refraction indices, say,  $d^i$ , are identical. That is

$$d^1 := h_{D^1}(\frac{\pi}{2}) + h_{D^1}(-\frac{\pi}{2}) = d^2. \quad (3.1)$$

Then, from (2.50) and (2.51),

$$h_{D^1}(\theta) = h_{D^2}(\theta), \forall \theta. \quad (3.2)$$

In particular, let  $N^i(a)$ ,  $i=1,2$ , be defined as in (2.32). Lemma 2.14 says

$$N^1(a) = N^2(a). \quad (3.3)$$

Let  $y^i(r, z)$  be the asymptotic solution given as in (2.27) with respect to index  $n^i(r)$ . Now we apply (3.3) with (2.27) to the following entire function of normal type,

$$F(z) := y^1(a, z) - y^2(a, z), \mathbb{C} \setminus 0i + \mathbb{R}, \quad (3.4)$$

so we have

$$h_F(\theta) = \limsup_{r \rightarrow \infty} \frac{\ln |y^1(a, re^{i\theta}) - y^2(a, re^{i\theta})|}{r} = 0, \theta \neq 0, \pi. \quad (3.5)$$

Since  $h_F(\theta)$  is continuous in  $\theta \in [0, 2\pi]$ , we have

$$h_F(\theta) \equiv 0. \quad (3.6)$$

Therefore, using Lemma 2.5, we conclude that  $F(z)$  is an exponential function of zero type.

As inspired by Aktosun, Gintides and Papanicolaou [1], McLaughlin and Polyakov [12], McLaughlin [13, 14], we seek to identify  $n(r)$  by inverse Sturm-Liouville theorem. In particular, we apply the results in [13].

**Theorem 3.1 (McLaughlin)** *We consider the following Sturm-Liouville problem*

$$z'' + (k^2 - q)z = 0, 0 < x < 1; \quad (3.7)$$

$$z(0) = z(1) = 0, \quad (3.8)$$

where  $q \in L^2(0, 1)$ . For another boundary condition,

$$z(0) = z'(1) = 0. \quad (3.9)$$

Suppose  $q_1, q_2 \in L^2(0, 1)$  and,  $\lambda_n(q_1) = \lambda_n(q_2)$ , the eigenvalues to (3.7) and (3.8),  $\mu_n(q_1) = \mu_n(q_2)$ , the eigenvalues to (3.7) and (3.9),  $\forall n \in \mathbb{N}$ . Then,  $q_1 \equiv q_2$ , a.e.

We need more lemmas to apply such a theorem. One is the following Phragmén-Lindelöf type of theorem in Levin [11, p.38, Theorem 3]:

**Lemma 3.2** *If  $f(z)$ ,  $z = x + iy$ , is an analytic function in the half-plane  $\{z | \Im z > 0\}$  such that, for all  $\epsilon > 0$ ,*

$$M_f(r) \underset{\text{as}}{<} e^{(\sigma + \epsilon)r},$$

*and  $|f(x)| \leq M$  on the real axis, then*

$$|f(x + iy)| \leq Me^{\sigma y}.$$

**Lemma 3.3** *Let  $n^i(r)$ ,  $i = 1, 2$ , be two refraction indices satisfying (1.1) generating same set of interior transmission eigenvalues in either  $\Sigma_1$  or  $\Sigma_2$  and  $n^1(0) = n^2(0)$ . Let  $y^i(a, z)$  be the solution defined by index  $n^i(r)$ . Then,  $y^1(a, z) \equiv y^2(a, z)$  and  $(y^1)'(a, z) - (y^2)'(a, z) \equiv 0$ .*

**Proof** The type of  $y^1(a, z) - y^2(a, z)$  as an entire function of exponential type is zero by Lemma 2.5. That is  $\sigma = 0$  for previous lemma. Moreover, given  $N^1(a) = N^2(a)$  and (2.26),

$$F(z) = y^1(a, z) - y^2(a, z) \rightarrow 0, \text{ as } z \rightarrow 0i \pm \infty. \quad (3.10)$$

Using Lemma 3.2 and Liouville's theorem in complex analysis, we conclude that  $y^1(a, z) - y^2(a, z) \equiv 0$ , for some constant. (3.10) again implies that  $y^1(a, z) - y^2(a, z) \equiv 0$ .

Similarly, from (2.29), we obtain  $(y^1)'(a, z) - (y^2)'(a, z) \equiv 0$ .  $\square$

Now we consider the Liouville transformation of (1.8):

$$z(\xi) := [n(r)]^{\frac{1}{4}} y_l(r), \text{ where } \xi := \int_0^r [n(\rho)]^{\frac{1}{2}} d\rho. \quad (3.11)$$

In this case, (1.8) becomes

$$\begin{cases} z'' + [k^2 - p(\xi)]z = 0; \\ z(0) = 0; z(N(a)) = y(a). \end{cases} \quad (3.12)$$

where

$$p(\xi) := \frac{n''(r)}{4[n(r)]^2} - \frac{5}{16} \frac{[n'(r)]^2}{[n(r)]^3} + \frac{2}{r^2 n(r)}. \quad (3.13)$$

The zeros of  $y(a, k)$  exactly corresponds to the eigenvalues of the Sturm-Liouville problem

$$\begin{cases} y'' + (k^2 n(r) - \frac{2}{r^2})y = 0, 0 < r < a; \\ y(0) = 0; y(a) = 0. \end{cases} \quad (3.14)$$

Similarly, the zeros of  $y'(a, k)$  exactly corresponds to the eigenvalues of the Sturm-Liouville problem

$$\begin{cases} y'' + (k^2 n(r) - \frac{2}{r^2})y = 0, 0 < r < a; \\ y(0) = 0; y'(a) = 0. \end{cases} \quad (3.15)$$

Under the transform (3.11), the zeros of  $z(N(a), k)$  exactly corresponds to the eigenvalues of the Sturm-Liouville problem

$$\begin{cases} z'' + [k^2 - p(\xi)]z = 0, 0 < \xi < N(a); \\ z(0) = 0; z(N(a)) = 0. \end{cases} \quad (3.16)$$

Similarly, the zeros of  $z'(N(a), k)$  exactly corresponds to the eigenvalues of the Sturm-Liouville problem

$$\begin{cases} z'' + [k^2 - p(\xi)]z = 0, 0 < \xi < N(a); \\ z(0) = 0; z'(N(a)) = 0. \end{cases} \quad (3.17)$$

Under transformation (3.11), the quantity  $z^i(N(a), k)$  and  $(z^i)'(N(a), k)$  corresponding to refraction index  $n^i(r)$  has common zeros. Hence, the Sturm-Liouville problems

$$\begin{cases} (z^i)'' + [k^2 - p^i(\xi)]z^i = 0, 0 < \xi < N(a); \\ z^i(0) = 0; z^i(N(a)) = 0, \end{cases} \quad (3.18)$$

have the same eigenvalues for both  $i = 1, 2$ . Similarly,

$$\begin{cases} (z^i)'' + [k^2 - p^i(\xi)]z^i = 0, 0 < \xi < N(a); \\ z^i(0) = 0; (z^i)'(N(a)) = 0. \end{cases} \quad (3.19)$$

have the same eigenvalues for both  $i = 1, 2$ . Given  $p^i \in \mathcal{L}^2$ ,  $i = 1, 2$ , Theorem 3.1 says these imply  $p^1 \equiv p^2$ . Therefore, we have

$$z^1(\xi, k) \equiv z^2(\xi, k), \forall \xi, k. \quad (3.20)$$

This says that

$$[n^1(r)]^{\frac{1}{4}} y^1(r, k) \equiv [n^2(r)]^{\frac{1}{4}} y^2(r, k), \forall r, k. \quad (3.21)$$

Since  $n^i$  never vanishes, the solutions  $y^1(r, k)$ ,  $y^2(r, k)$  have common zero set in  $\mathbb{C}$  of its maximal density as described by Cartwright's theory. From (2.27), their density is  $\frac{N^i(r)}{\pi}$  which can be computed using (2.41) and (2.42). Hence,

$$\int_0^r [n^1(\rho)]^{\frac{1}{2}} d\rho = \int_0^r [n^2(\rho)]^{\frac{1}{2}} d\rho, \forall r. \quad (3.22)$$

We have  $n^1 \equiv n^2$  by assumption (1.1).

## References

- [1] T. Aktosun, D. Gintides and V.G. Papanicolaou, The uniqueness in the inverse problem for transmission eigenvalues for the spherically symmetric variable-speed wave equation, *Inverse Problems*, v.27, 115004(2011).
- [2] M.L. Cartwright, On the directions of Borel of functions which are regular and of finite order in an angle, *Proc. London Math. Soc. ser.2* vol.38, 503-541(1933).
- [3] M.L. Cartwright, *Integral functions*, Cambridge University Press, 1956.
- [4] D. Colton and R. Kress, *Inverse acoustic and electromagnetic scattering theory*, 2nd ed. Applied mathematical science, v.93, Springer-Verlag, 1998.
- [5] D. Colton, P. Monk and Jiguang Sun, Analytical and computational methods for transmission eigenvalues, *Inverse problems*, v.26, 045011(2010).
- [6] D.G. Dickson, Zeros of exponential sums, *Proc. Amer. Math. Soc.*, 16, 84-89(1965).
- [7] D.G. Dickson, Expansions in series of solutions of linear difference-differential and infinite order differential equations with constant coefficients, *Memoirs of the American Mathematical Society*, Rhode Island, USA, No.23, 1957.
- [8] A. Erdelyi, *Asymptotics expansions*, Dover, 1956.
- [9] R.E. Langer, On the zeros of exponential sums and integrals, *Bull. Amer. Math. Soc.*, 37, No.4, 213-239(1931).
- [10] B. Ja. Levin, *Distribution of zeros of entire functions*, revised edition, *Translations of mathematical monographs*, American mathematical society, 1972.
- [11] B. Ja. Levin, *Lectures on entire functions*, Translation of mathematical monographs, V.150, AMS, 1996.
- [12] J.R. McLaughlin and P.L. Polyakov, On the uniqueness of a spherically symmetric speed of sound from transmission eigenvalues, *Jour. Differential Equations*, 107, 351-382(1994).
- [13] J.R. McLaughlin, P.E. Sacks, M. Somasundaram, *Inverse scattering in acoustic media using interior transmission eigenvalues*, *Inverse problems in wave propagation* (Minneapolis, MN, 1995), 357-374, IMA Vol. Math. Appl., 90, Springer, New York, 1997.
- [14] J.R. McLaughlin, Inverse spectral theory using nodal points as data-a uniqueness result, *Jour. Differential Equations*, 73, 354-362(1988).